Oral Exam, Applied and Computational Mathematics

Question I

Consider the Allen-Cahn equation for $u := u(x,t) \in \mathbb{R}$ (natural boundary conditions)

$$\partial_t u = \partial_{xx} u + u(1-u^2), \quad u(x,0) = u_0(x), \quad x \in (0,1), \quad t > 0,$$

where u_0 is a sufficiently smooth function.

1. Denote the energy functional as $E(u)$
$$E(u(\cdot,t)) = \int_0^1 \left(\frac{1}{2}|\partial_x u(x,t)|^2 + \frac{1}{4}(u(x,t)-1)^2 \right) dx.$$
Show $E(u(\cdot,t))$ is a decreasing function respect to t.

2. Consider a semi-discrete-in-time scheme
$$\frac{u^{n+1}-u^n}{\tau} = \partial_{xx} u^{n+1} - (u^{n+1})^3 + u^n, \tag{1}$$
where $u^0 = u_0$ and u^n is the numerical approximation of $u(\cdot, t_n)$ with $t_n = n\tau$ ($\tau > 0$ is the time step size, $n = 0, 1, 2, \cdots$). Show that the scheme is energy stable, i.e. $E(u^{n+1}) \le E(u^n)$ ($n = 0, 1, 2, \cdots$).

3. For given u^n, prove that (1) admits a unique solution u^{n+1} at each time step.

Question II

For an integer $n \ge 1$, consider the equation

$$f(x) = 0, f(x) = x^{n+1} - b^n x + ab^n, a > 0, b > 0.$$

(a) Prove that the equation has exactly two distinct positive roots if and only if
$$a < \frac{n}{(n+1)^{1+\frac{1}{n}}} b.$$

(b) Assuming that the condition in (a) holds, show that Newton's method converges to the smaller positive root, when started at $x_0 = a$, and to the larger one, when started at $x_0 = b$.

Question III

1. Consider a non-singular tridiagonal $n \times n$ matrix given by:

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}$$

with no zero on all three diagonals. Denote the strictly lower triangular, diagonal, strictly upper triangular of A by L, D, U respectively.

(a) Let $B_J = I_n - D^{-1}A$, and $p_{B_J}(\lambda)$ be the characteristic polynomial of B_J. Also, let $B_{GS} = -(L+D)^{-1}U$, and $p_{B_{GS}}(\lambda)$ be the charateristic polynomial of B_{GS}. Prove that

$$p_{B_J}(\lambda) = \det(-D^{-1})\det(L + \lambda D + U) \text{ and}$$

$$p_{B_{GS}}(\lambda) = \det(-(L+D)^{-1})\det(\lambda L + \lambda D + U).$$

(b) Prove that

$$\det(\lambda^2 L + \lambda^2 D + U) = \lambda^n \det(L + \lambda D + U).$$

(c) Using part (a) and (b), prove that

$$\rho(B_{GS}) = \rho(B_J)^2,$$

where $\rho(B_{GS})$ and $\rho(B_J$ are the spectral radii of B_{GS} and B_J respectively. Hence, determine whether Jacobi or Gauss-Seidel methods to solve $A\mathbf{x} = \mathbf{b}$ converges faster if they both converge. Please explain your answers with details.